Phys 410
Fall 2014
Lecture #21 Summary
11 November, 2014

We considered the motion of a "top" or gyroscope that was set into motion at angular velocity ω along one of its principal axes and then supported at a single point on its rotation axis. The top is rotating about one of its principal axes, which we will call the 3-axis, with direction $\widehat{e_3}$. The top is observed to precess in a cone around the vertical direction $\hat{z}$. We can write the angular momentum as $\vec{L} = \lambda_3 \omega \widehat{e_3}$, where λ_3 is the principal moment for this axis. There are two forces acting on the top, the normal force at the point of support, and the weight, acting on the center of mass. We take the origin to be at the point of support so that only the weight exerts a torque. The torque leads to a time rate of change of the angular momentum: $\vec{\Gamma} = \dot{\vec{L}}$. The torque is $\vec{\Gamma} = \vec{R} \times M\vec{g}$, which points in a direction perpendicular to $\widehat{e_3}$, and therefore $\vec{L}$. This means that $|\vec{L}|$ remains fixed, but the direction of $\vec{L}$ will change. We found that $\dot{\widehat{e_3}} = \vec{\Omega} \times \widehat{e_3}$, where $\vec{\Omega} = \frac{RMg}{\lambda_3 \omega} \hat{z}$, showing that the principal axis of the top $\widehat{e_3}$ is rotating around the $\hat{z}$ axis at angular velocity $\frac{RMg}{\lambda_3 \omega}$. This is the rate of precession. From the demonstration we saw that as the gyroscope winds down (ω decreases), the rate of precession increases, consistent with this result.

We then considered the description of Newton's second law from the perspective of an observer on the rotating object. The observer in the "body frame" can identify the principal axes of the object and describe the angular momentum using the diagonalized inertia tensor as $\vec{L} = (\lambda_1 \omega_1, \lambda_2 \omega_2, \lambda_3 \omega_3)$. An inertial observer in the "space frame" is in position to identify correctly the net torque $\vec{\Gamma}$ acting on the angular momentum vector, and to write Newton's second law of motion (in rotational form) as $\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{space}$. We learned how to translate the time derivative of a vector quantity from an inertial frame to a rotating reference frame in Lecture 10: $\left(\frac{d\vec{Q}}{dt}\right)_{Space} = \left(\frac{d\vec{Q}}{dt}\right)_{Body} + \vec{\Omega} \times \vec{Q}$, where $\vec{Q}$ is the vector in question and the non-inertial reference frame is rotating with angular velocity $\vec{\Omega}$. In this case we can write the equations of motion as witnessed in the body frame as $\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{Body} + \vec{\omega} \times \vec{L}$, which translates in component form into the Euler equations:

$$\Gamma_1 = \lambda_1 \dot{\omega}_1 - \omega_2 \omega_3 (\lambda_2 - \lambda_3)$$

$$\Gamma_2 = \lambda_2 \dot{\omega}_2 - \omega_1 \omega_3 (\lambda_3 - \lambda_1)$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 - \omega_1 \omega_2 (\lambda_1 - \lambda_2)$$

This set of equations describes how the angular velocity vector evolves as it is acted upon by a net external torque. The hard part of using these equations is taking the torque from the space frame and expressing it in component form in the body frame (i.e. Γ_1, Γ_2, Γ_3). When applied to the case of the spinning top discussed above, we note that $\Gamma_3 = 0$ (the torque acts in a direction perpendicular to $\hat{e_3}$) and $\lambda_1 = \lambda_2$, hence $\lambda_3 \dot{\omega}_3 = 0$, so that ω_3 is constant. Thus the angular velocity vector remains aligned with 3-axis and no other component of $\vec{\omega}$ is excited.

To simplify things further we take the special case of rotational motion under torque-free conditions, $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$. An application of these equations was to the case of an object rotating about one principal axis ($\hat{e}_3$), but then given a small kick to produce rotations about the other axes ($\hat{e}_1$). The analysis led to a simple equation of motion: $\ddot{\omega}_1 = -\left[\frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2} \omega_3^2\right] \omega_1$, where it was found that ω_3 is approximately constant. This yields simple-harmonic motion (SHM) for ω_1 and ω_2 as a function of time if λ_3 is either the largest or smallest of the three principal moments. SHM means that the motion about the original axis is stable and the motion about the other axes oscillates around zero. If λ_3 is the middle eigenvalue, then the square-bracket term is negative, yielding a solution for $\omega_1(t)$ that grows exponentially in time. This is a sign of instability. We demonstrated this phenomenon with a book, where motion around the principal axes with largest and smallest moments was fairly stable, while motion about the third (middle moment of inertia) was clearly much less stable. The video from the ISS showing torque-free rotational motion of the Russian-English dictionary illustrated this quite clearly.

Finally we derived the equation of motion that describes nutation of a precessing gyroscope in a gravitational field. First we introduced the Euler angles. This is a convention that describes the orientation of an object based on a combination of both the space frame and the body frame defined by the principal axes of the object. Starting with both coordinate systems aligned ($\hat{x} \leftrightarrow \hat{e}_1$, etc.) and the origins coincident, first rotate by an angle φ around the $\hat{z}$ axis. Next rotate about the new $\hat{e}_2{}'$ axis by and angle θ to create the final $\hat{e}_3$ direction. Finally rotate by and angle ψ about the $\hat{e}_3$ axis. This process is simply a convention for how to align an arbitrary rigid body relative to its center of mass.

With the Euler angles, we can now express the angular velocity and angular kinetic energy in terms of these angles and their time derivatives. In other words we can say $\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$ and for the kinetic energy $T = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}(\lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2)$, which is written in terms of the body frame. If we translate the expression for the rotational kinetic energy in to the Euler angles, after some vector algebra, we get, $T = \frac{1}{2}\lambda_1\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$. This expression assumes a cylindrically symmetric object (e.g. a gyroscope) with $\lambda_1 = \lambda_2$. The potential energy of the gyroscope is simply $U = MgR\cos\theta$, where M is the total mass of the gyroscope, R is the location of the center of mass from the point of support, and g is the acceleration due to gravity. The Lagrangian for the gyroscope is just $\mathcal{L} = T - U$, and one sees immediately that both ϕ and ψ are cyclic (or ignorable) coordinates, hence their conjugate momenta p_ϕ and p_ψ are constants of the motion. These constants are equal to the z-component and the 3-component of the angular momentum vector. The remaining θ equation is a one-dimensional second order differential equation for a "particle" that lives in a limited-range effective potential $U_{eff}(\theta)$ with total energy $E = T + U_{eff}(\theta)$ given by $E = \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{eff}(\theta)$ and $U_{eff}(\theta) = \frac{\left(p_\phi - p_\psi \cos\theta\right)^2}{2\lambda_1 \sin^2\theta} + \frac{{p_\psi}^2}{2\lambda_3} + MgR\cos\theta$. The effective potential diverges at the two limiting values of θ, namely 0 and π, and forms a minimum in between. The "θ-particle" with finite total energy E therefore bounces back and forth between two classical turning points in θ, which represent the limits of the "nodding" up and down, which is the phenomenon known as nutation.